

# SU(5) Symmetry of *spdfg* Interacting Boson Model

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## Abstract

The extended interacting boson model with *s*-, *p*-, *d*-, *f*- and *g*-bosons being included (*spdfg* IBM) are investigated. The algebraic structure including the generators, the Casimir operators of the groups at the SU(5) dynamical symmetry and the branching rules of the irreducible representation reductions along the group chain are obtained. The typical energy spectrum of the symmetry is given.

**Key Words:** Extended interacting boson model, Octupole deformation, Superdeformation, Shape coexistence

## 1 Introduction

It has been well known that the interacting boson model (IBM)[1], is successful in describing the spectroscopic properties of low-lying nuclear levels that contain quadrupole collectivity and pairing effects. In this theory, the pairs of valence nucleons (particles/holes) are approximated as bosons. Then the configuration space is decreased dramatically with respect to that in shell model. Furthermore, it has been documented that collective motion is closely related to the dynamical symmetries of the system. For example, in the original version of IBM, which includes only *s*- and *d*-bosons and those of protons and neutrons are not distinguished from each other, vibration corresponds to U(5) symmetry, axial rotation to SU(3) symmetry and  $\gamma$ -unstable rotation to O(6) symmetry[1]. We can then implement algebraic method to study the dynamical properties of nuclei.

Recently, the nuclear states with octupole deformation which displays a reflection asymmetric shape have been one of the focus in nuclear structure studies[2, 3]. It has been shown that, with the *p*- and *f*-bosons being included, the interacting boson model (*spdf* IBM) can describe the nuclear states with both quadrupole and octupole deformations well[4, 5, 6, 7, 8, 9]. On the other hand, to describe the nuclear states with large quadrupole moment and hexadecupole deformation, the *g*-bosons should be taken into account[10, 11, 12]. And hence the *sdg* IBM was put forward and the algebraic structure has been investigated well[13, 14]. Meanwhile, it has also been shown that the *sdg* IBM is quite powerful to describe superdeformed (SD) states with positive parity[15, 16, 17, 18]. The SD states are those at the extreme condition with large deformation and very high rotational frequency, and built upon the second well of the nuclear energy surface. Up to now, several fascinating phenomena, such as the identical bands,  $\Delta I = 4$  bifurcation, which challenge the traditional nuclear many-body theory, have been observed in SD states. With the IBM being extended to include *s*-, *p*-, *d*-, *f*-, and *g*-bosons, and to the supersymmetry formalism, not only the global properties of the positive parity and negative parity SD bands but also the identical bands, the  $\Delta I = 4$  bifurcation can be described well[17, 19, 20, 21, 22, 23, 24]. Even though the *spdfg* IBM is so successful in describing nuclear states, only the SU(3) symmetry has been investigated thoroughly[25] up to now. Analyzing the algebraic structure of the *spdfg* IBM in more detail, one can know that there exists another dynamical symmetry, namely the SU(5) symmetry, which has been shown to be more suitable to describe SD states[17, 19, 20, 21, 22, 23, 24]. Then we will discuss the SU(5) symmetry of the *spdfg* IBM in this paper.

## 2 Dynamical symmetries of the *spdfg* IBM

In the *spdfg* IBM, where the *p*-, *f*- and *g*-bosons are included alongside *s*- and *d*-bosons, the space spanned by the single boson states is  $\sum_l (2l+1) = 1 + 3 + 5 + 7 + 9 = 25$  dimensional. The largest symmetry group is then  $U(25)$ . The generators of the group can be expressed as the Racah tensors

$$G_q^k(l'l') = (b_l^\dagger \tilde{b}_{l'})_q^k = \sum_{\mu\mu'} \langle l\mu l'\mu' | kq \rangle b_{l\mu}^\dagger \tilde{b}_{l'\mu'}, \quad (1)$$

where  $b_{l\mu}^\dagger$  is the creation operator of the boson with angular momentum  $\{l, \mu\}$ ,  $\tilde{b}_{l\mu} = (-1)^{l+\mu} b_{l-\mu}$  is the spherical tensor conjugate to  $b_{l\mu}^\dagger$  with  $b_{l\mu}$  being the annihilation operator of the boson with angular momentum  $\{l, \mu\}$ .

To describe the dynamical symmetries of nuclear states consisting of the *s*-, *p*-, *d*-, *f*- and *g*-bosons, we should consider group chains starting from  $U(25)$  and ending at  $SO(3)$  due to the conservation of angular momentum. It is evident that there exist maximal subgroups  $U_s(1) \otimes U_{pdfg}(24)$ ,  $U_{sp}(4) \otimes U_{dfg}(21)$ ,  $U_{sd}(6) \otimes U_{pfg}(19)$ ,  $U_{sf}(8) \otimes U_{pdg}(17)$ ,  $U_{sg}(10) \otimes U_{pdf}(15)$ ,  $U_{spd}(9) \otimes U_{fg}(16)$ ,  $U_{spf}(11) \otimes U_{dg}(14)$ ,  $U_{sdg}(15) \otimes U_{pf}(10)$ , and so on. Because *s*-, *d*- and *g*-bosons hold positive parity and *p*-, *f*-bosons possess negative parity, and nuclear states are usually classified with their parity, we can concentrate our discussion to the largest subgroup  $U_{sdg}(15) \otimes U_{pf}(10)$ .

To discuss the dynamical symmetry group chains, one should find the subgroups of  $U_{sdg}(15)$  and  $U_{pf}(10)$  at first, then couple the ones at same level with each other. It has been known that the group  $U_{sdg}(15)$  has subgroups  $SU_{sdg}(6)$ ,  $SU_{sdg}(5)$  and  $SU_{sdg}(3)$  [13, 14], and the group  $U_{pf}(10)$  holds subgroups  $SU_{pf}(5)$  and  $SU_{pf}(3)$  [5, 6, 7]. These subgroups can couple with each other at levels  $SU(5)$  and  $SU(3)$ . Therefore, the *spdfg* IBM possesses dynamical symmetries  $SU(5)$  and  $SU(3)$ . The  $SU(3)$  symmetry has been investigated in Ref.[25]. We discuss then the  $SU(5)$  symmetry in the *spdfg* IBM in this work.

## 3 Algebraic Structure of the $SU_{spdfg}(5)$ Symmetry

On the coupling level  $SU_{sdg}(5) \otimes SU_{pf}(5) \supset SU_{spdfg}(5)$ , we have the dynamical group chain

$$\begin{aligned} U_{spdfg}(25) &\supset U_{sdg}(15) \otimes U_{pf}(10) \supset SU_{sdg}(5) \otimes SU_{pf}(5) \supset SU_{spdfg}(5) \\ &\supset SO_{spdfg}(5) \supset SO_{spdfg}(3), \end{aligned} \quad (2)$$

The wavefunction can be expressed in terms of the irreducible representations (irreps) of the groups as:

$$|\psi_L^T\rangle = |N; n_{sdg}, [n_i^+]_5; n_{pf}, [n_i^-]_5; \gamma [n_i]_5; \beta (\nu_1, \nu_2)_5; \alpha L\rangle \quad \begin{matrix} SU_{sdg}(5) & SU_{pf}(5) & SU_{spdfg}(5) & SO_{spdfg}(5) & SO_{spdfg}(3) \end{matrix} \quad (3)$$

where  $[n_i^+]_5$ ,  $[n_i^-]_5$  and  $[n_i]_5$  denote the irreps  $[n_1^+, n_2^+, n_3^+, n_4^+]_5$ ,  $[n_1^-, n_2^-, n_3^-, n_4^-]_5$ ,  $[n_1, n_2, n_3, n_4]_5$  of the groups  $SU_{sdg}(5)$ ,  $SU_{pf}(5)$  and  $SU_{spdfg}(5)$ , respectively.  $(\nu_1, \nu_2)$  is the irrep of the group  $SO_{spdfg}(5)$ .  $L$  is the irrep of the  $SO_{spdfg}(3)$ , i.e., the angular momentum of the system.  $\beta$  is the additional quantum number to distinguish the same  $(\nu_1, \nu_2)_5$  belong to the same  $[n_1, n_2, n_3, n_4]_5$ , and  $\alpha$  is the additional quantum number to distinguish the same  $L$  belong to the same  $(\nu_1, \nu_2)_5$ .  $\gamma$  is the multiplicity of the irrep  $[n_1, n_2, n_3, n_4]$ .

The structure of subgroup and the branching rules for each step of the irrep reductions are the following.

(1)  $U(25) \supset U_{sdg}(15) \otimes U_{pf}(10)$

With the general principle of the boson realization of unitary group, we have the generators of the group  $U_{sdg}(15)$  as

$$B(l, l')_q^k = [b_l^\dagger \tilde{b}_{l'}]_q^{(k)}, \quad l, l' = 0, 2, 4 \quad (4)$$

The generators of the group  $U_{pf}(10)$  can be given as

$$F(l, l')_q^k = [b_l^\dagger \tilde{b}_{l'}]_q^{(k)}, \quad l, l' = 1, 3 \quad (5)$$

The branching rule to reduce the irrep  $[N]_{25}$  of the  $U(25)$  to the irreps of  $U_{sdg}(15) \otimes U_{pf}(10)$  is

$$[N]_{25} = \sum_{\oplus} [n_{sdg}]_{15} \otimes [n_{pf}]_{10}, \quad (6)$$

where  $n_{sdg}$  and  $n_{pf}$  are the possible non-negative integers satisfying the relation  $n_{sdg} + n_{pf} = N$ . It is obvious that  $n_{sdg}$  is the number of bosons with positive parity and  $n_{pf}$  is the negative parity bosons' number, and  $N$  is the total number of the bosons.

**(2)  $U_{sdg}(15) \supset SU_{sdg}(5)$**

It has been known that the system consisting of particles with angular momentum  $\lambda$  holds the symmetry  $U(2\lambda + 1)$ . Meanwhile, for the system possessing the  $U(N)$  symmetry with  $N = \sum_l 2l + 1$ , if a system including  $r$  virtual identical particles with angular momentum  $\lambda$  can couple to angular momentum set  $\{l\}$ , the group  $SU(2\lambda + 1)$  is a subgroup of  $U(\sum_l 2l + 1)$ , and the generators of the subgroup  $SU(2\lambda + 1)$  can be given as[13]

$$\hat{T}_q^k = \sum_{ll'} \sqrt{\frac{2l+1}{2k+1}} \langle \lambda^r l || [a_\lambda^\dagger \tilde{a}_\lambda]^k || \lambda^r l' \rangle [b_l^\dagger \tilde{b}_{l'}]_q^k \quad (7)$$

Because the system with two  $\lambda = 2$  particles can form the angular momenta  $l = 0, 2, 4$ , the s-, d-, g-boson system has then the subgroup  $SU_{sdg}(5)$ . From Eq.(7), one can get the generators of this subgroup as

$$\hat{Q}_q^{(1)}(sdg) = [d^\dagger \tilde{d}]^{(1)} + \sqrt{6} [g^\dagger \tilde{g}]^{(1)}, \quad (8)$$

$$\hat{Q}_q^{(2)}(sdg) = \frac{2\sqrt{5}}{5} [s^\dagger \tilde{d} + d^\dagger \tilde{s}]^{(2)} - \frac{3}{7} [d^\dagger \tilde{d}]^{(2)} + \frac{12\sqrt{5}}{35} [d^\dagger \tilde{g} + g^\dagger \tilde{d}]^{(2)} + \frac{3\sqrt{22}}{7} [g^\dagger \tilde{g}]^{(2)}, \quad (9)$$

$$\hat{Q}_q^{(3)}(sdg) = \frac{8}{7} [d^\dagger \tilde{d}]^{(3)} - \frac{3\sqrt{10}}{7} [d^\dagger \tilde{g} + g^\dagger \tilde{d}]^{(3)} - \frac{3}{7} \sqrt{11} [g^\dagger \tilde{g}]^{(3)}, \quad (10)$$

$$\hat{Q}_q^{(4)}(sdg) = \frac{2\sqrt{5}}{5} [s^\dagger \tilde{g} + g^\dagger \tilde{s}]^{(4)} + \frac{4}{7} [d^\dagger \tilde{d}]^{(4)} + \frac{\sqrt{110}}{7} [d^\dagger \tilde{g} + g^\dagger \tilde{d}]^{(4)} + \frac{\sqrt{715}}{35} [g^\dagger \tilde{g}]^{(4)}, \quad (11)$$

where  $\alpha = \pm 1$ ,  $\beta = \pm 1$ , the construct constants of Lie group are independent of the sign of  $\alpha$  or  $\beta$ .

The branching rule for the reduction  $U_{sdg}(15) \supset U_{sdg}(5)$  is[13]

$$[n_{sdg}]_{15} = \sum_{q,r,s,t} [2n_{sdg} - 4q - 6r - 8s - 10t, 2r + 2s + 2t, 2s + 2t, 2t] \quad (12)$$

where  $s, t, q, r$  are integers satisfying

$$2n_{sdg} - 4q - 6r - 8s - 10t \geq 2r + 2s + 2t \geq 2s + 2t \geq 2t \geq 0$$

**(3)  $U_{pf}(10) \supset SU_{pf}(5)$**

Along the same way to handle the  $SU_{sdg}(5)$  group, we get the generators of group  $SU_{pf}(5)$  as

$$\hat{Q}_q^{(1)}(pf) = \frac{\sqrt{5}}{5} [p^\dagger \tilde{p}]_q^{(1)} + \frac{\sqrt{70}}{5} [f^\dagger \tilde{f}]_q^{(1)}, \quad (13)$$

$$\hat{Q}_q^{(2)}(pf) = -\frac{\sqrt{21}}{5} [p^\dagger \tilde{p}]_q^{(2)} + \frac{2\sqrt{6}}{5} [p^\dagger \tilde{f} + f^\dagger \tilde{p}]_q^{(2)} + \frac{\sqrt{6}}{5} [f^\dagger \tilde{f}]_\mu^{(2)}, \quad (14)$$

$$\hat{Q}_q^{(3)}(pf) = \frac{\sqrt{30}}{5} [p^\dagger \tilde{f} + f^\dagger \tilde{p}]_q^{(3)} - \frac{\sqrt{15}}{5} [f^\dagger \tilde{f}]_q^{(3)}, \quad (15)$$

$$\hat{Q}_q^{(4)}(pf) = -\frac{\sqrt{10}}{5} [p^\dagger \tilde{f} + f^\dagger \tilde{p}]_q^{(4)} - \frac{\sqrt{55}}{5} [f^\dagger \tilde{f}]_q^{(4)}. \quad (16)$$

The branching rule of the reduction  $U_{pf}(10) \supset SU_{pf}(5)$  can be given as[5, 6, 7]

$$[n_{pf}]_{10} = \begin{cases} [n_{pf}, n_{pf}, 0, 0] \oplus [n_{pf}-1, n_{pf}-1, 1, 1] \oplus [n_{pf}-2, n_{pf}-2, 2, 2] \oplus \dots \\ \dots \oplus [\frac{1}{2}n_{pf}, \frac{1}{2}n_{pf}, \frac{1}{2}n_{pf}, \frac{1}{2}n_{pf}] & n_{pf} \text{ is even} \\ [n_{pf}, n_{pf}, 0, 0] \oplus [n_{pf}-1, n_{pf}-1, 1, 1] \oplus [n_{pf}-2, n_{pf}-2, 2, 2] \oplus \dots \\ \dots \oplus [\frac{1}{2}(n_{pf}+1), \frac{1}{2}(n_{pf}+1), \frac{1}{2}(n_{pf}-1), \frac{1}{2}(n_{pf}-1)] & n_{pf} \text{ is odd} \end{cases} \quad (17)$$

(4)  $SU_{sdg}(5) \otimes SU_{pf}(5) \supset SU_{spdfg}(5)$

It is obvious that the generators of  $SU_{spdfg}(5)$  are

$$\hat{T}_q^{(k)}(spdfg) = \hat{Q}_q^{(k)}(sdg) + \hat{Q}_q^{(k)}(pf), \quad (18)$$

where  $\hat{Q}_q^{(k)}(sdg)$  and  $\hat{Q}_q^{(k)}(pf)$  are generators of the groups  $SU_{sdg}(5)$ ,  $SU_{pf}(5)$ , respectively.

The reduction  $SU_{sdg}(5) \otimes SU_{pf}(5) \supset SU_{spdfg}(5)$  obeys *Littlewood* rule and can be given with the Young tableaux technique[26].

(5)  $SU_{spdfg}(5) \supset SO_{spdfg}(5)$

The generators of the group  $SO_{spdfg}(5)$  contain the tensors of rank 1 and 3, they are identified as the odd rank tensors of the  $SU_{spdfg}(5)$

$$\hat{T}^{(1)} = \sqrt{\frac{1}{5}}[p^\dagger \tilde{p}]^{(1)} + [d^\dagger \tilde{d}]^{(1)} + \sqrt{\frac{14}{5}}[f^\dagger \tilde{f}]^{(1)} + \sqrt{6}[g^\dagger \tilde{g}]^{(1)} \quad (19)$$

$$\begin{aligned} \hat{T}^{(3)} = & [d^\dagger \tilde{d}]^{(3)} - \frac{3}{8}\sqrt{11}[g^\dagger \tilde{g}]^{(3)} - \alpha \frac{3}{4}\sqrt{\frac{5}{2}}[d^\dagger \tilde{g} + g^\dagger \tilde{d}]^{(3)} - \sqrt{\frac{3}{5}}[f^\dagger \tilde{f}]^{(3)} \\ & - \sqrt{\frac{6}{5}}[p^\dagger \tilde{f} + f^\dagger \tilde{p}]^{(3)} \end{aligned} \quad (20)$$

Using the Young tableaux technique[26, 27] or the Schur function method[28], the branching rule of this reduction can be fixed.

(6)  $SO_{spdfg}(5) \supset SO_{spdfg}(3)$

The generators of the group  $SO_{spdfg}(3)$  are

$$\hat{L}_q^{(1)} = \sqrt{10}\hat{T}_q^{(1)}, \quad (21)$$

where  $\hat{T}_q^{(1)}$  are the generators of the group  $SO_{spdfg}(5)$ .

The irrep reduction of  $SO(5) \supset SO(3)$  is not simple. However, it has been discussed in detail. All the branching rules can be obtained by referring to computer calculation or tables[26].

Ending this section, we give all the Casimir operators that associate with the  $SU(5)$  symmetry in table 1.

## 4 Energy Spectrum of the SU(5) Symmetry

For a nucleus with the SU(5) symmetry in the *spdfg* IBM, the Hamiltonian can be written as

$$\begin{aligned} H = & \epsilon C_{1U_{sdg}(15)} + \epsilon' C_{1U_{pf}(10)} + A C_{2SU_{sdg}(5)} + A' C_{2SU_{pf}(5)} + B C_{2SU_{spdfg}(5)} \\ & + C C_{2SO_{spdfg}(5)} + D C_{2SO_{spdfg}(3)} \end{aligned} \quad (22)$$

Table 1: Casimir operators of U(25) and its subgroups

Group	IRRP	Casimir	Eigenvalue
$U_{sdgpf}$	$[N]$	$C_{1U(25)} = \sum_{l=0}^4 b_l^+ b_l = \hat{N}$	$N$
$U_{sdg}(15)$	$[n_{sdg}]$	$C_{1U_{sdg}(15)} = [s^+ \tilde{s}] + \sqrt{5}[d^+ \tilde{d}]_0^0 + 3[g^+ \tilde{g}]_0^0$ $C_{2U_{sdg}(15)} = \sum_{l,l',k} B(l,l')^k \cdot B(l,l')^k$	$n_{sdg}$ $n_{sdg}(n_{sdg} + 14)$
$U_{pf}(10)$	$[n_{pf}]$	$C_{1U_{pf}(10)} = \sqrt{3}[p^+ \tilde{p}]_0^0 + \sqrt{7}[f^+ \tilde{f}]_0^0$ $C_{2U_{pf}(10)} = \sum_{l,l',k} F(l,l')^k \cdot F(l,l')^k$	$n_{sdg}$ $n_{sdg}(n_{sdg} + 9)$
$SU_{sdg}(5)$	$[n_1^+, n_2^+, n_3^+, n_4^+]$	$C_{2SU_{sdg}(5)} = \sum_k \hat{Q}^{(k)}(sdg) \cdot \hat{Q}^{(k)}(sdg)$	$f(n_1^+, n_2^+, n_3^+, n_4^+)^*$
$SU_{pf}(5)$	$[n_1^-, n_2^-, n_3^-, n_4^-]$	$C_{2SU_{pf}(5)} = \sum_k \hat{Q}^{(k)}(pf) \cdot \hat{Q}^{(k)}(pf)$	$f(n_1^-, n_2^-, n_3^-, n_4^-)^*$
$SU_{spdfg}$	$[n_1, n_2, n_3, n_4]$	$C_{2SU_{spdfg}(5)} = \sum_k \hat{T}^{(k)} \cdot \hat{T}^{(k)}$	$f(n_1, n_2, n_3, n_4)^*$
$SO_{spdfg}(5)$	$(\nu_1, \nu_2)$	$C_{2SO_{spdfg}(5)} = 2(\hat{T}^{(1)} \cdot \hat{T}^{(1)} + \hat{T}^{(3)} \cdot \hat{T}^{(3)})$	$\nu_1(\nu_1 + 3) + \nu_2(\nu_2 + 1)$
$SO_{sdgpf}(3)$	$L$	$C_{2SO_{sdgpf}(3)} = 2(\hat{L} \cdot \hat{L})$	$L(L + 1)$

\*where  $f(x_1, x_2, x_3, x_4) = x_1(x_1 + 4) + x_2(x_2 + 2) + x_3^2 + x_4(x_4 - 2) - \frac{1}{5}(x_1 + x_2 + x_3 + x_4)^2$

The energy of the state can be expressed in terms of the irreps of the groups as

$$\begin{aligned}
 E(L^\pi) = & E_0 + \epsilon n_{sdg} + \epsilon' n_{pf} \\
 & + A[n_1^+(n_1^+ + 4) + n_2^+(n_2^+ + 2) + n_3^{+2} + n_4^+(n_4^+ - 2) - \frac{1}{5}(n_1^+ + n_2^+ + n_3^+ + n_4^+)^2] \\
 & + A'[n_1^-(n_1^- + 4) + n_2^-(n_2^- + 2) + n_3^{-2} + n_4^-(n_4^- - 2) - \frac{1}{5}(n_1^- + n_2^- + n_3^- + n_4^-)^2] \\
 & + B[n_1(n_1 + 4) + n_2(n_2 + 2) + n_3^2 + n_4(n_4 - 2) - \frac{1}{5}(n_1 + n_2 + n_3 + n_4)^2] \\
 & + C[\nu_1(\nu_1 + 3) + \nu_2(\nu_2 + 1)] + DL(L + 1)
 \end{aligned} \tag{23}$$

To give an example explicitly, we discuss a SU(5) symmetry nucleus with six valence nucleons, i.e., with boson number  $N = 3$ . According to the branching rules of the irrep reductions along the group chain, the quantum numbers of all the possible states can be listed in Table 2.

With a chosen set of parameters  $\epsilon = 0.3$  MeV,  $\epsilon' = 0.7$  MeV,  $A = A' = 0$ ,  $B = -0.06$  MeV,  $C = 0.05$  MeV,  $D = 0.006$  MeV, we get the energy spectra with positive, negative parity as shown in Fig. 1, Fig. 2, respectively.

Looking through the figures, one can easily know that much more energy bands than those in the *sd* IBM and the *spdf* IBM appear here and the band structure is more complicated than that in the previous models. Only for the subset with positive parity, there exist vibrational bands with angular momentum sequence  $\Delta I = 4$  belonging to the totally symmetric irrep [6] of the SU(5) group. Meanwhile, the non-totally symmetric irrep [4, 2] generates a rotational bands with angular momentum sequence  $\Delta I = 2$ . In this band, the states with angular momentum  $I = 4k$  ( $k$  is an integer) are generated by the totally symmetric irreps of the SO(5) group, while those with  $I = 4k + 2$  are generated by the non-totally symmetric irreps of the SO(5). With the definition of the transition energy  $E_\gamma(I) = E(I) - E(I - 2)$ , one can readily get the contribution of the term with the SO(5) symmetry to the  $\gamma$ -ray energy  $E_\gamma$  as  $E_\gamma^{SO(5)}([I] = 4k + 2) = 6C$  and  $E_\gamma^{SO(5)}([I] = 4k) = (4[\frac{I}{2}] - 4)C$ . Therefore the term with the SO(5) symmetry makes the  $E_\gamma$  staggering even though the  $|C|$  may be very small. There involves then  $\Delta I = 2$  energy staggering in such a band. The rotational bands with  $\Delta I = 4$  bifurcation in both ND states and SD states may thus be attributed to that the nuclei may possess the  $SU_{spdfg}(5)$  symmetry. On the other hand, combining the spectrum with positive parity and that with negative parity, one may know that there display bands with angular momentum sequence  $2^+, 3^-, 4^+, 5^-, \dots$ , or  $2^-, 3^+, 4^-, 5^+, \dots$ , which

Table 2: Examples of the branching rules of the irreducible representation reduction in the symmetry. Part A. positive parity states

$U_{sdg}(15) \otimes U_{pf}(10)$	$SU_{sdg}(5) \otimes SU_{pf}(5)$	$SU_{spdfg}(5)$	O(5)	O(3)
$[n_{sdg}] \otimes [n_{pf}]$	$[n^+] \otimes [n^-]$	$[n]$	$(\nu_1, \nu_2)$	$L$
$[3] \otimes [0]$	$[6] \otimes [0]$	$[6]$	(6, 0)	0,3,4,6 <sup>2</sup> ,7,8,9,10,12
			(4, 0)	2,4,5,6,8
			(2, 0)	2,4
			(0, 0)	0
	$[42] \otimes [0]$	$[42]$	(4, 2)	0,1,2 <sup>2</sup> ,3 <sup>2</sup> ,4 <sup>3</sup> ,5 <sup>2</sup> ,6 <sup>3</sup> ,7 <sup>2</sup> ,8 <sup>2</sup> ,9,10
			(4, 0)	2,4,5,6,8
			(3, 1)	1,2,3 <sup>2</sup> ,4,5 <sup>2</sup> ,6,7
			(2, 2)	0,2,3,4,6
			(2, 0) <sup>2</sup>	2,4
			(0, 0)	0
	$[222] \otimes [0]$	$[222]$	(2, 2)	0,2,3,4,6
			(2, 0)	2,4
			(0, 0)	0
$[1] \otimes [2]$	$[2] \otimes [22]$	$[42]$	(4, 2)	0,1,2 <sup>2</sup> ,3 <sup>2</sup> ,4 <sup>3</sup> ,5 <sup>2</sup> ,6 <sup>3</sup> ,7 <sup>2</sup> ,8 <sup>2</sup> ,9,10
			(4, 0)	2,4,5,6,8
			(3, 1)	1,2,3 <sup>2</sup> ,4,5 <sup>2</sup> ,6,7
			(2, 2)	0,2,3,4,6
			(2, 0) <sup>2</sup>	2,4
			(0, 0)	0
		$[321]$	(3, 2)	1,2,3,4,5,6,7,8
			(3, 1)	1,2,3 <sup>2</sup> ,4,5 <sup>2</sup> ,6,7
			(2, 2)	0,2,3,4,6
			(2, 1)	1,2,3,4,5
		$[222]$	(2, 0)	2,4
			(1, 1)	1,3
			(2, 2)	0,2,3,4,6
	$[2] \otimes [1111]$	$[3111]$	(2, 0)	2,4
			(0, 0)	0
			(3, 0)	0,3,4,6
		$[2]$	(2,1)	1,2,3,4,5
			(1, 0)	2
			(2, 0)	2,4
			(0, 0)	0

(Table 2 continued) Part B. negative parity states

$U(15) \otimes U(10)$	$SU_{sdg}(5) \otimes SU_{pf}(5)$	$SU_{spdfg}(5)$	O(5)	O(3)
$[n_{sdg}] \otimes [n_{pf}]$	$[n]^+ \otimes [n]^-$	$[n]$	$(\nu_1, \nu_2)$	$L$
$[2] \otimes [1]$	$[4] \otimes [11]$	[51]	(5, 1)	1,2,3 <sup>2</sup> ,4 <sup>2</sup> ,5 <sup>3</sup> ,6 <sup>2</sup> ,7 <sup>3</sup> ,8 <sup>2</sup> ,9 <sup>2</sup> ,10,11
			(4, 0)	2,4,5,6,8
			(3, 1)	1,2,3 <sup>2</sup> ,4,5 <sup>2</sup> ,6,7
			(2, 0)	2,4
			(1, 1)	1,3
		[411]	(4, 1)	1,2,3 <sup>2</sup> ,4 <sup>2</sup> ,5 <sup>2</sup> ,6 <sup>2</sup> ,7 <sup>2</sup> ,8,9
			(3, 1)	1,2,3 <sup>2</sup> ,4,5 <sup>2</sup> ,6,7
			(2, 1)	1,2,3,4,5
			(1, 1)	1,3
			(3, 3)	1,3 <sup>2</sup> ,4,5,6,7,9
	$[22] \otimes [11]$	[33]	(3, 1)	1,2,3 <sup>2</sup> ,4,5 <sup>2</sup> ,6,7
			(1, 1)	1,3
			(3, 2)	1,2 <sup>2</sup> ,3,4 <sup>2</sup> ,5 <sup>2</sup> ,6,7,8
		[321]	(3, 1)	1,2,3 <sup>2</sup> ,4,5 <sup>2</sup> ,6,7
			(2, 2)	0,2,3,4,6
			(2, 1)	1,2,3,4,5
			(2, 0)	2,4
			(1, 1)	1,3
		[2211]	(2, 1)	1,2,3,4,5
			(1, 1)	1,3
$[0] \otimes [3]$	$[0] \otimes [33]$	[33]	(3, 3)	1,3 <sup>2</sup> ,4,5,6,7,9
			(3, 1)	1,2,3 <sup>2</sup> ,4,5 <sup>2</sup> ,6,7
			(1, 1)	1,3
	$[0] \otimes [2211]$	[2211]	(2, 1)	1,2,3,4,5
			(1, 1)	1,3

is usually referred to as “simplex symmetry”. The appearance of such kind bands is just the feature of the nucleus with both octupole deformation and quadrupole, hexadecupole deformations.

## 5 Summary and Remarks

In this paper, the  $SU(5)$  dynamical symmetry of the  $spdfg$  IBM has been discussed. The algebraic structure such as the generators, Casimir operators, branching rules of the irreducible representation reductions, and the typical energy spectrum are obtained. It shows that many bands that do not exist in the  $sd$  IBM and  $spdf$  IBM appear in the  $SU_{spdfg}(5)$  symmetry and the band structure is more complicated. Of particular interest are the existence of both vibrational bands generated by the totally symmetric irrep of the  $SU_{spdfg}(5)$  group and rotational bands generated by the non-totally symmetric irrep. The bands with  $\Delta I = 4$  bifurcation in the normally deformed states and superdeformed states may be attributed to that the nuclei may hold the  $SU_{spdfg}(5)$  symmetry. Moreover, the simultaneous appearance of the vibrational bands and rotational bands indicates that the  $SU(5)$  symmetry in  $spdfg$  IBM may be applicable to describe the shape coexistence and phase transition in nuclei.

On the other hand, it is known that electromagnetic transition rates are important signatures of nuclear structure. In general, with only the one-body operators being taken into account, the E2, E1, E3, M1 transition (which are believed to be the more important ones) operator can be given as

$$\begin{aligned} \hat{T}(E2) = & e_2 \{ [s^\dagger \tilde{d} + d^\dagger \tilde{s}]^{(2)} + \chi_{dd}^2 [d^\dagger \tilde{d}]^{(2)} + \chi_{gg}^2 [g^\dagger \tilde{g}]^{(2)} + \chi_{dg}^2 [d^\dagger \tilde{g} + g^\dagger \tilde{d}]^{(2)} \\ & + \chi_{pp}^2 [p^\dagger \tilde{p}]^{(2)} + \chi_{ff}^2 [f^\dagger \tilde{f}]^{(2)} + \chi_{pf}^2 [p^\dagger \tilde{f} + f^\dagger \tilde{p}]^{(2)} \}, \end{aligned} \quad (24)$$

$$\hat{T}(E1) = e_1 \{ [s^\dagger \tilde{p} + p^\dagger \tilde{s}]^{(1)} + \chi_{pd}^1 [p^\dagger \tilde{d} + d^\dagger \tilde{p}]^{(1)} + \chi_{df}^1 [d^\dagger \tilde{f} + f^\dagger \tilde{d}]^{(1)} + \chi_{fg}^1 [f^\dagger \tilde{g} + g^\dagger \tilde{f}]^{(1)} \}, \quad (25)$$

$$\hat{T}(E3) = e_3 \{ [s^\dagger \tilde{f} + f^\dagger \tilde{s}]^{(3)} + \chi_{pd}^3 [p^\dagger \tilde{d} + d^\dagger \tilde{p}]^{(3)} + \chi_{df}^3 [d^\dagger \tilde{f} + f^\dagger \tilde{d}]^{(3)} + \chi_{fg}^3 [f^\dagger \tilde{g} + g^\dagger \tilde{f}]^{(3)} \}, \quad (26)$$

$$\hat{T}(M1) = g_L \hat{L}, \quad (27)$$

where  $\chi_{\alpha\beta}^k(\alpha, \beta = p, d, f, g)$  are the structure constants,  $e_k(k = 1, 2, 3)$  are the effective charge,  $\hat{L}$  is the angular momentum operator as shown in Eq. (21),  $g_L$  is the rotational g-factor. After analysis, the selection rules can be expressed as  $\Delta n_{sdg} = 0$  with  $\Delta n_d = \pm 1, 0$  and/or  $\Delta n_g = \mp 1, 0$  or  $\Delta n_{pf} = 0$  with  $\Delta n_p = 0, \pm 1$  and/or  $\Delta n_f = 0, \mp 1$  for the E2 transition. The selection rules for the E1 and E3 transitions can be given as  $\Delta n_{sdg} = \pm 1$  with  $\Delta n_s = \pm 1$  and/or  $\Delta n_d = \pm 1$  and/or  $\Delta n_g = \pm 1$  and  $\Delta n_{pf} = \mp 1$  with  $\Delta n_p = \mp 1$  and/or  $\Delta n_f = \mp 1$ . However the calculation of the reduced transition rates are so complicated that we postpone it now.

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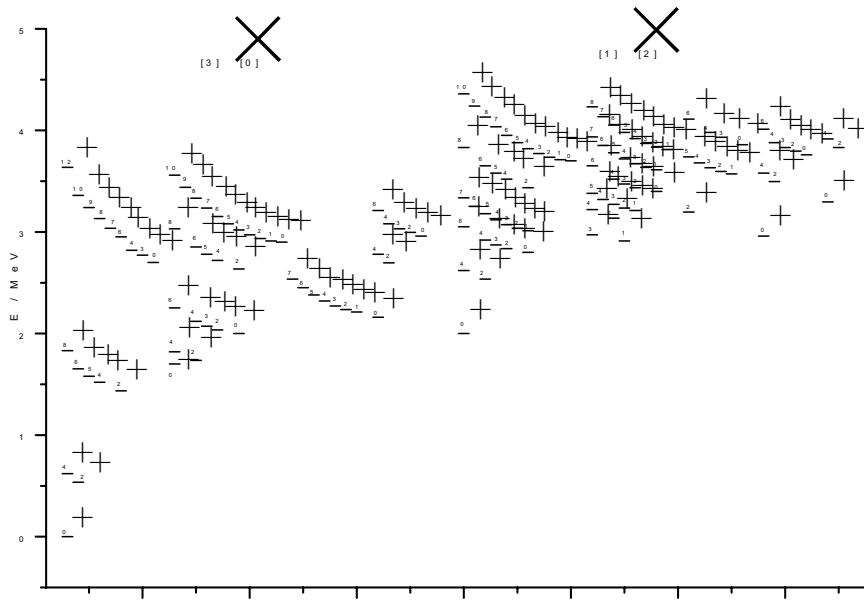


Figure 1: Energy spectrum of the positive parity states with total boson number  $N = 3$